Partial Derivatives with Constrained Variables

When we compute partial derivatives of $w=f(x,y)$, we assume x and y are independent variables.

However, it is not always the case.

e.g. Suppose the internal energy U of a gas may be expressed as a function

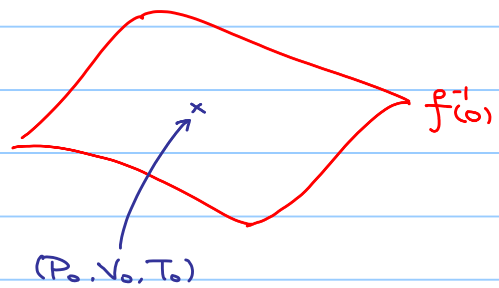
$U = F(P,V,T)$, where P = pressure, V = volume, T = temperature.

But P, V, T are not independent variables since

$$PV = nRT \quad \text{(ideal gas law)}$$

where n, R are constants.

Let $f(P,V,T) = PV - nRT$,



Around (P_0, V_0, T_0), there are three possible cases:

① P depends on V, T

② V depends on P, T

③ T depends on P, V

There is no ambiguity to talk about $\frac{\partial P}{\partial V}$ and etc.

But, ...

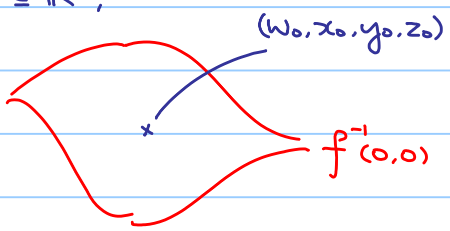
e.g. Find $\frac{\partial w}{\partial x}$ if $w = x^2+y^2+z^2$ and $z = x^2+y^2$.

What does it mean ?

$\frac{\partial w}{\partial x} \rightsquigarrow$ w depends on x.

Let $f(w,x,y,z) = (w-x^2-y^2-z^2,\ x^2+y^2-z)$

Consider $f^{-1}(0,0) \subseteq \mathbb{R}^4$,



Around (w_0, x_0, y_0, z_0), there are two possible cases:

① w, y depend on x, z

② w, z depend on x, y

For ① : x and z are independent variables.

Sub $y^2 = z - x^2$ into $w = x^2+y^2+z^2$,

$w = x^2 + (z-x^2) + z^2 = z + z^2$

so $\frac{\partial w}{\partial x} = 0$

For ② : x and y are independent variables

Sub $z = x^2+y^2$ into $w = x^2+y^2+z^2$,

$w = x^2+y^2+(x^2+y^2)^2 = x^4+2x^2y^2+y^4+x^2+y^2$

so $\frac{\partial w}{\partial x} = 4x^3+4xy^2+2x$

Which one do we refer ?

Notations .

$\left(\frac{\partial w}{\partial x}\right)_y$ means finding $\frac{\partial w}{\partial x}$ with x and y independent.

$\left(\frac{\partial w}{\partial x}\right)_z$ means finding $\frac{\partial w}{\partial x}$ with x and z independent.

e.g. Find $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ if $w = x^2+y-z+\sin t$ and $t = x+y$

$$w = x^2+y-z+\sin t$$
$$= x^2+y-z+\sin(x+y)$$
$$\left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x+0-0+\cos(x+y)\cdot\frac{\partial}{\partial x}(x+y)$$
$$= 2x+\cos(x+y)$$

e.g. If $f(x,y,z)=0$, show that $\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1$.

Suppose x depends on y and z.

$$f(x(y,z),y,z) = 0$$

(Take $\frac{\partial}{\partial y}$) $$\frac{\partial f}{\partial x}\cdot\left(\frac{\partial x}{\partial y}\right)_z + \frac{\partial f}{\partial y} = 0$$

$$\therefore \left(\frac{\partial x}{\partial y}\right)_z = -f_y/f_x$$

Similarly, $\left(\frac{\partial y}{\partial z}\right)_x = -f_z/f_y$ and $\left(\frac{\partial z}{\partial x}\right)_y = -f_x/f_z$

$$\therefore \left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1.$$